MATH 2050 C Lecture 9 (Feb 15)

Recall: $\lim(x_n) = x \in \mathbb{R}$ iff (ε-K def$^{\underline{n}}$)

$$\boxed{\forall \varepsilon > 0,\ \exists K \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K}$$

A seq. (x_n) is said to be convergent if such an $x \in \mathbb{R}$ exists (hence unique!). Otherwise, divergent.

Question: (1) When does $\lim(x_n)$ exist?

(2) If exists, how to compute the limit?

§ Limit Theorems (§ 3.2)

indep. of n

Def$^{\underline{n}}$: (x_n) is bounded if $\exists M > 0$ s.t.

$$|x_n| \leq M \quad \forall n \in \mathbb{N}$$

Remark: This is the same as saying that the subset

$$\{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R} \text{ is bounded.}$$

Thm: (x_n) convergent $\Rightarrow$ (x_n) bounded

Remark: The converse "$\Leftarrow$" is false.

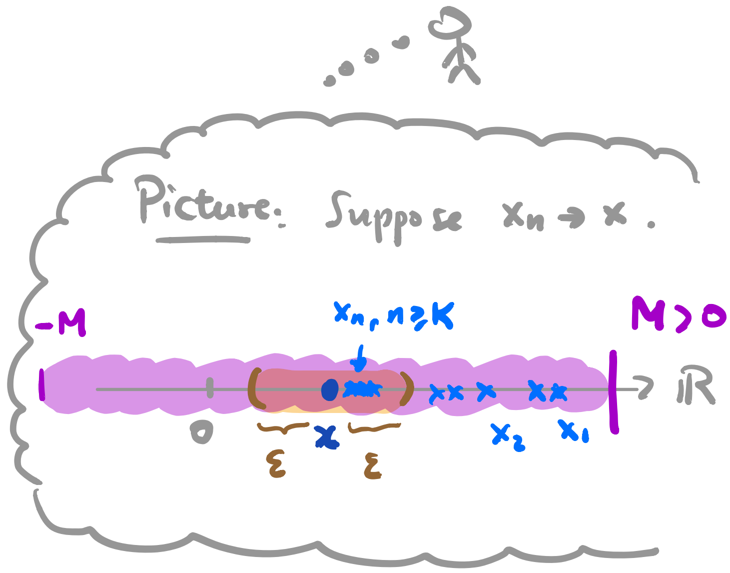
E.g. $(x_n) = ((-1)^n)$ is divergent but bdd.

Cor: (x_n) unbdd $\Rightarrow$ (x_n) divergent

Remark: This is useful in proving a seq. is divergent

E.g.) The seq. $(x_n) = (n)$ is unbdd, hence must be divergent.

Proof of Thm:



Since (x_n) is convergent, by def^{n}, $\exists\, x \in \mathbb{R}$ s.t.

$$\lim (x_n) = x$$

i.e. $\forall \varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \quad \forall n \geq K.$$

Take $\varepsilon = 1$, then $\exists\, K = K(1) \in \mathbb{N}$ s.t.

$$|x_n - x| < 1 \quad \forall n \geq K$$

By Triangle ineq. $\forall n \geq K$

$$|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| < 1 + |x|$$

Choose $M := \max\{|x_1|, |x_2|, \dots, |x_{K-1}|, 1+|x|\} > 0$

$$\Rightarrow \quad |x_n| \leq M \quad \forall n \in \mathbb{N}$$

$\blacksquare$

Recall: $\mathbb{R}$ is the complete ordered field.

Q: How is the limiting process compatible with these structures?

Limit Theorems:

Suppose $\lim(x_n) = x$ and $\lim(y_n) = y$. Then:

(i) $\lim(x_n \pm y_n) = x \pm y$

(ii) $\lim(x_n y_n) = xy$

!(iii)! $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$ provided $y_n \neq 0 \ \forall n \in \mathbb{N}$ and $y \neq 0$

Remark: The limits exist and are equal to the "expected" value.

Proof: We will prove (i) – (iii) one by one.

$\boxed{\text{(i)}}$ **Claim:** $\boxed{\lim(x_n + y_n) = x + y}$

 ooo

Goal: Find K s.t. $|(x_n + y_n) - (x+y)| < \varepsilon \quad \forall n \geq K$

$$\text{L.H.S.} \leq \underbrace{|x_n - x|}_{\substack{\text{small} \\ \because x_n \to x}} + \underbrace{|y_n - y|}_{\substack{\text{small} \\ \because y_n \to y}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Pf of Claim: Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim(x_n) = x$ and $\lim(y_n) = y$ by assumption,

$\exists\, K_1, K_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2} \qquad \forall n \geq K_1$$

and

$$|y_n - y| < \frac{\varepsilon}{2} \qquad \forall n \geq K_2$$

Choose $K := \max\{K_1, K_2\} \in \mathbb{N}$, then $\forall n \geq K$,

$$|(x_n + y_n) - (x+y)| \overset{\Delta\text{-ineq.}}{\leq} |x_n - x| + |y_n - y|$$

$$\underset{\because K \geq K_1, K_2}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(i) $\square$

$\boxed{\text{(ii)}}$ Claim: $\boxed{\lim(x_n y_n) = xy}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Since (y_n) is convergent, by previous Thm, (y_n) is bdd

ie $\exists\, M > 0$ s.t.

$$|y_n| \leq M \quad \forall n \in \mathbb{N}$$

Goal: $\exists\, K \in \mathbb{N}$ st $\forall n \geq K$
$|x_n y_n - xy| < \varepsilon$

"Estimate" the LHS.

$$|x_n y_n - xy|$$

$$= |x_n y_n - xy_n + xy_n - xy|$$

$$\leq \underbrace{|y_n|}_{\text{bdd}} \cdot \underbrace{|x_n - x|}_{\text{small}} + \underbrace{|x|}_{\text{fixed number}} \cdot \underbrace{|y_n - y|}_{\text{small}}$$

$$\leq M\,|x_n - x| + M \cdot |y_n - y|$$

Take $M' := \max\{M, |x|\} > 0$.

By def^{n} of limit, consider the error $\varepsilon/2M' > 0$.

$\exists\, K_1, K_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon/2M' \qquad \forall n \geq K_1$$

and $$|y_n - y| < \varepsilon/2M' \qquad \forall n \geq K_2$$

Choose $K := \max\{K_1, K_2\} \in \mathbb{N}$ st $\forall n \geq K$,

$$|x_n y_n - xy| = |x_n y_n - xy_n + xy_n - xy|$$

$$= |y_n(x_n - x) + x(y_n - y)|$$

$$\leq |y_n| \cdot |x_n - x| + |x| \cdot |y_n - y|$$

$$\leq M \cdot |x_n - x| + |x| \cdot |y_n - y|$$

$$\leq M' \cdot |x_n - x| + M' \cdot |y_n - y|$$

$$< M' \cdot \frac{\varepsilon}{2M'} + M' \cdot \frac{\varepsilon}{2M'} = \varepsilon$$

(ii) $\square$

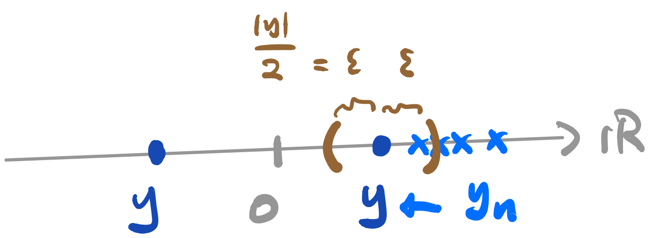
(iii) Claim: $\boxed{\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}}$ assume $y \neq 0$, $y_n \neq 0$ $\forall n \in \mathbb{N}$

Observe: $\left(\frac{x_n}{y_n}\right) = \left(x_n \cdot \frac{1}{y_n}\right)$, by (ii), it suffices to prove

$(*)$: $\boxed{\lim\left(\frac{1}{y_n}\right) = \left(\frac{1}{y}\right)}$ provided $y \neq 0$ and $y_n \neq 0 \ \forall n \in \mathbb{N}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Q: Is $\frac{1}{|y_n|}$ bdd ?



Goal: $\exists K \in \mathbb{N}$ st $\forall n \geq K$

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \varepsilon$$

Estimate the L.H.S.

$$\left|\frac{1}{y_n} - \frac{1}{y}\right|$$

$$= \left|\frac{y_n - y}{y_n y}\right|$$

$$= \underbrace{\boxed{\frac{1}{|y_n|}}}_{\text{bdd ?}} \cdot \underbrace{\frac{1}{|y|}}_{\substack{\text{fixed} \\ \text{bdd} \\ \because y \neq 0}} \cdot \underbrace{|y_n - y|}_{\substack{\text{small} \\ \because y_n \to y}}$$

We prove a lemma first.

Lemma: $\exists \tilde{K} \in \mathbb{N}$ s.t.

$$|y_n| \geq \frac{|y|}{2} \quad \forall n \geq \tilde{K}$$

Pf of Lemma: Since $\lim(y_n) = y$, consider $\frac{|y|}{2} > 0$ ($\because y \neq 0$)

$\Rightarrow \exists \tilde{K} \in \mathbb{N}$ st $|y_n - y| < \frac{|y|}{2} \quad \forall n \geq \tilde{K}$.

Then, $\forall n \geq \tilde{K}$,

$$|y_n| = |y + (y_n - y)| \overset{\text{reversed } \Delta\text{-ing}}{\geq} \big||y| - |y_n - y|\big| \geq \frac{|y|}{2}$$

Since $\lim(y_n) = y$, taking $\frac{\varepsilon/2}{|y|^2} > 0$

$\exists\, K' \in \mathbb{N}$ s.t. $|y_n - y| < \frac{\varepsilon/2}{|y|^2} \quad \forall n \geq K'$

Choose $K := \max\{\tilde{K}, K'\} \in \mathbb{N}$, then $\forall n \geq K$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y_n - y}{y_n y}\right| = \frac{1}{|y_n|} \cdot \frac{1}{|y|} \cdot |y_n - y|$$

$$< \underbrace{\frac{1}{|y|/2}}_{\substack{\text{by Lemma} \\ \because k \geq \tilde{K}}} \cdot \frac{1}{|y|} \cdot \underbrace{\frac{\varepsilon/2}{|y|^2}}_{\because k \geq K'} = \varepsilon$$

(iii)

$\square$

Remark: The assumptions in (iii) are necessary.

E.g. $(y_n) = \left(\frac{1}{n}\right) \to y = 0$

But $\left(\frac{1}{y_n}\right) = (n)$ is divergent.

Remark: The "converse" is NOT true in general.

E.g. Consider $(x_n) = \left(\frac{1}{n}\right)$, $(y_n) = (n)$.

although $(x_n y_n) = \left(\frac{1}{n} \cdot n\right) = (1) \to 1$.

but (y_n) is divergent!

Thm: Let $(x_n), (y_n)$ be two convergent seq. s.t.

$$x_n \leq y_n \quad \forall n \in \mathbb{N} \quad \cdots\cdots (\dagger)$$

THEN: $\lim(x_n) \leq \lim(y_n)$.

Remark: Even if we assume that the inequality is strict in $(\dagger)$, ie

$$x_n < y_n \quad \forall n \in \mathbb{N}$$

then we can still only conclude that

$$\lim(x_n) \leq \lim(y_n)$$

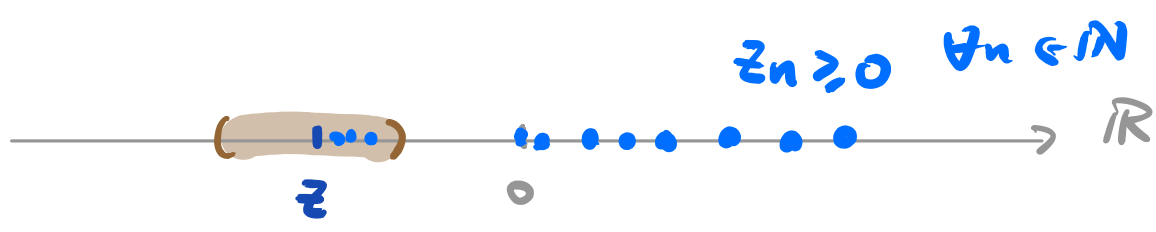
E.g.) $0 < \frac{1}{n} \quad \forall n \in \mathbb{N}$ BUT $0 = \lim\left(\frac{1}{n}\right)$

Proof: By Limit Thm. it suffices to show

Claim: (z_n) convergent seq. s.t. $z_n \geq 0 \quad \forall n \in \mathbb{N}$

$$\Longrightarrow \quad \lim(z_n) =: z \geq 0.$$

Picture:



Proof by contradiction: Suppose NOT, then $z < 0$.

Then. $|z| > 0$. Consider $\varepsilon := \frac{|z|}{2} > 0$,

by def^{n} of limit, $\exists K \in \mathbb{N}$ s.t.

$$|z_n - z| < \varepsilon = \frac{|z|}{2} \quad \forall n \geq K.$$

$$\Rightarrow \quad z_n < z + \frac{|z|}{2} = -\frac{|z|}{2} < 0 \quad \forall n \geq K.$$

This contradicts $z_n \geq 0 \ \forall n \in \mathbb{N}$.

$\square$